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Second-Order Characterizable Cardinals and Ordinals

Abstract. The notions of finite and infinite second-order characterizability of cardinal and ordinal numbers are developed. Several known results for the case of finite characterizability are extended to infinite characterizability, and investigations of the second-order theory of ordinals lead to some observations about the Fraenkel-Carnap question for well-orders and about the relationship between ordinal characterizability and ordinal arithmetic. The broader significance of cardinal characterizability and the relationships between different notions of characterizability are also discussed.

Keywords: Second-order Logic, Cardinal Characterizability, Ordinal Characterizability, Fraenkel-Carnap Question

AMS classification: 03B15 and 03C85

1. Background, Relevance, and Notions of Characterizability

The information that a logic can express about the cardinality of a domain is among the most important measures of its expressive power. The upward and downward Löwenheim-Skolem theorems, considered to be among the most important results of first-order model theory, pertain directly to the indistinguishability of domains of different cardinalities, and in [6], Montague used cardinal characterizability as an important standard by which to distinguish the expressive power of different higher-order logics. The investigation of which cardinals are characterizable in second-order logic is thus an important area of inquiry within second-order model theory. The serious study of second-order cardinal characterizability was begun by Garland in [3] and [4], and related ideas have subsequently been approached from a variety of perspectives (e.g. [7], [8], and [9]). Ordinal characterizability was also introduced in [3], and was shown to be intimately connected with cardinal characterizability. This paper considers two different notions of cardinal characterizability, along with the corresponding notions of ordinal characterizability, and investigates their properties, with an eye toward extending some of Garland's more important results.

This paper is concerned mainly with the accustomed polyadic second-order logics (hereafter simply "second-order logics"), which allow quantification over individual, unary predicate, relational, and functional variables, using the "standard" interpretations for all second-order quantifiers. For any non-logical vocabulary $\mathbf{k}$, let $L_{2(\mathbf{k})}$ be the second-order logic with vocabulary $\mathbf{k}$, and let $L_{2M(\mathbf{k})}$ be the *monadic* second-order logic with vocabulary $\mathbf{k}$ (that is, the fragment of $L_{2(\mathbf{k})}$ with only individual and unary predicate variables). The discussion focuses on the *pure* second-order logic L_{2P}, with no non-logical vocabulary, and on $L_{2(Q)}$ and $L_{2M(Q)}$, second-order logics (the latter monadic) over the non-logical vocabulary consisting of a single binary relation symbol Q. Naturally, any nonempty set is an interpretation of L_{2P}, although no generality is lost by speaking only of models whose domains are nonzero cardinal numbers. $L_{2(Q)}$ and $L_{2M(Q)}$, meanwhile, have interpretations of the form $\mathfrak{A} = (A, \rho)$, where A is a nonempty set and ρ is a relation on A. However, since it is possible to express an axiom of well-ordering in monadic second-order logic, it will often be possible and convenient to speak as if the only models under consideration are the strict well-orders associated with particular ordinal numbers. In the discussion that follows, the standard distinction between sentences and formulas is used.

A structure $\mathfrak{A}$ of signature $\mathbf{k}$ is *(second-order) infinitely characterizable* iff there is Φ, a (possibly infinite) set of sentences of $L_{2(\mathbf{k})}$, such that $\mathfrak{A} \models \Phi$ and all models of Φ are isomorphic to $\mathfrak{A}$. $\mathfrak{A}$ is *monadic second-order infinitely characterizable* iff there is such a Φ containing only sentences of $L_{2M(\mathbf{k})}$. $\mathfrak{A}$ is *(second-order) finitely characterizable* if some such Φ is finite, and *monadic second-order finitely characterizable* if such a Φ is both finite and made of sentences from $L_{2M(\mathbf{k})}$. For finitely characterizable structures, I shall always assume that Φ contains exactly one sentence φ. No loss of generality results, since φ can be obtained by conjoining the finitely many sentences in Φ.

For any cardinal number $\kappa > 0$, (κ) is the structure consisting entirely of the domain κ, and to speak of the characterizability of κ (in either of the above senses) is to speak of the characterizability of (κ). Where $\alpha > 0$ is an ordinal $(\alpha, \in)$ is the strict well-order $(\alpha, \in\restriction \alpha)$; following [3] and [4], I shall use the term "ordinal characterizability" when talking about the characterizability of strict well-orders . Observe that $\kappa > 0$ is infinitely characterizable iff, for every nonzero cardinal $\lambda \neq \kappa$, there is φ such that $(\kappa) \models \varphi$ and $(\lambda) \not\models \varphi$, and that the analogous statement holds for infinitely characterizable ordinals. Most notions of cardinal characterizability discussed in the literature are versions of finite characterizability. The infinitely characterizable cardinals are those cardinals which are second-order equivalent only to themselves, and so are coextensive with Weaver's "minimal cardinals" (cf. [9]). Since there is a tradition of research in cardinal indistinguishability (see, e.g. [7]), this connection gives infinite characterizability a relevance independent of its interest as a generalization of finite characterizability.

Two other notions of expressive power over cardinals are closely related to finite characterizability. First, for any second-order sentence φ, the *spectrum* of φ is the class of all cardinal numbers $\mu > 0$ such that φ has at least one model of cardinality μ. A cardinal κ is finitely characterizable iff there is some second-order sentence φ such that κ is the only element of the spectrum of φ. The φ in question need not be pure: it can contain non-logical vocabulary of any type corresponding to a variable type of the logic, since one can replace all constants with existentially quantified variables without changing the the spectrum of φ. The idea of a spectrum plays an important role in [3] and [4]. Second, a cardinal κ is *(second-order) describable* iff there is a pure formula $\psi(S)$, with a unary predicate variable as its only free variable, such that, for every $\mathfrak{A}$ and every $\chi \subseteq A$, $\mathfrak{A} \vDash \psi(S)[\chi]$ iff χ is of cardinality κ (in this case $\psi(S)$ *describes* κ). It is easy to show (see Section 6) that a nonzero cardinal is describable iff it is finitely characterizable.

An important topic related to characterizability is the Fraenkel-Carnap question (see [2], [11], and [12]). A class of structures Δ has the *(second-order) Fraenkel-Carnap property* iff, for every structure $\mathfrak{A}$ in Δ, $\mathfrak{A}$ is finitely characterizable iff the second-order theory of $\mathfrak{A}$ is *finitely axiomatizable*, in the sense that there is some second-order sentence φ such that $\mathfrak{A} \vDash \varphi$ and for every second-order sentence ψ, if $\mathfrak{A} \vDash \psi$ then $\varphi \vDash \psi$. The *monadic second-order Fraenkel-Carnap property* is defined analogously, except that attention is restricted to monadic second-order sentences. The *Fraenkel-Carnap question* for a particular class of structures, with respect to a particular logic, is the question of whether that class of structures has the Fraenkel-Carnap property associated with that logic.

Cardinal characterizability is relevant to the study of many sorts of structures. In foundational issues related to categoricity, fixing the cardinality of the models of a theory can be seen as an important first step. More broadly, the cardinality of a structure's domain is often one of its more important properties, and it is interesting to see how much a second-order logic can say about this property. Just as it is an interesting property of second-order logic that it can state an axiom of well-ordering, define ancestral relations, or express a genuine axiom of infinity, so is it interesting that it can convey a great deal of fine-grained information about the cardinality of a model. The connection between finite characterizability and cardinal describability means that observations about finite characterizability also provide insights into what can be said about the size of certain structurally relevant subsets of a model (such as bases and generating sets of various algebraic structures, or the equivalence classes of an equivalence relation), so finite characterizability provides us with a variety of highly versatile descriptive tools for talking about many sorts of structures.

The primary motivation for the study of ordinal characterizability is its relevance to cardinal characterizability, but it is of interest for other reasons as well. Since well-orders are relatively simple and well understood structures, they serve as an ideal place to try out new techniques in second-order model theory, and may provide a useful measure of expressive power. Like other simple structures, they may also be a useful source of counterexamples to conjectures in this field. The second-order model theory of well-orders is also interesting in its own right, since well-orders are considered mathematically important.

This paper follows in the footsteps of [3] and [4], and several of the results below will look quite familiar from those works. There are, however, several important differences. Garland considered only finite characterizability by $\Diamond^1_2$ sentences, while the present treatment considers both finite and infinite characterizability by any sentences of L_{2P} or $L_{2(Q)}$. The proofs here include several other departures from the treatment in [3] and [4], the most significant of which are that they place a much stronger emphasis on the explicit construction of the sentences involved, and that the expressive power of second-order logic is considered not just in terms of the truth-conditions of sentences, but in terms of the satisfaction-conditions of the open formulas from which those sentences are built. A pleasant byproduct of these proofs is a toolbox of useful formulas that can be called into service to prove further results about cardinal characterizability, both in L_{2P} and in other languages, such as the infinitary second-order logics of [13]. For characterizable ordinals, this paper takes an entirely different tack from [3] and [4]: it gives countable ordinals much less attention, but provides an affirmative answer to the second-order and monadic second-order Fraenkel-Carnap questions for well-orders, and explores the relationship between the ordinal characterizability and the operations of ordinal arithmetic. Ordinal characterizability in $L_{2M(Q)}$ is considered in all these discussions, but is absent from [3] and [4].

One of the major contributions of [3] and [4] was to identify a fundamental connection between cardinal and ordinal characterizability; Section 2 demonstrates a variety of results about this connection. Section 3 turns to ordinal characterizability and establishes that the strict well-orders have the Fraenkel-Carnap property with respect to both $L_{2(Q)}$ and $L_{2M(Q)}$. To the knowledge of the author, this is the first proof of a nontrivial monadic second-order Fraenkel-Carnap property. Section 4 considers two known results about the closure of the finitely characterizable cardinals under operations of cardinal arithmetic, and extends them to infinitely characterizable cardinals. Section 5 returns to the ordinals and investigates the relationship between ordinal characterizability and ordinal arithmetic. Here, as in 3, the monadic case is also considered. Section 6 explores the relationships between some of the different notions of characterizability found in this paper and in the literature. Finally, Section 7 discusses open questions and directions for further research.

2. Relating the Second-Order Theories of Cardinals and Ordinals

It is shown in [3] and [4] that if $(\alpha, \in)$ is finitely characterizable by $\Diamond^1_2$ sentences, then so is $\aleph_\alpha$, and that the α^{th} element of any suitable spectrum is likewise finitely characterizable. In [8] (page 105), a similar result for finite characterizability is briefly discussed. Corollaries 2.1, 2.2, 2.3, and 2.4 below establish results of this kind for finite and infinite second-order characterizability.

The proofs of Corollaries 2.1 and 2.2 both depend on a construction that, for every sentence φ of $L_{2(Q)}$, provides a sentence $\aleph_\varphi$ of L_{2P} such that all models of $\aleph_\varphi$ are uncountable and, for all ordinals $\alpha > 0$, $(\alpha, \in) \models \varphi$ iff $(\aleph_\alpha) \models \aleph_\varphi$.

To see the significance of $\aleph_\varphi$ for cardinal characterizability, suppose that $(\alpha, \in)$ is finitely characterizable, and let φ be a sentence of $L_{2(Q)}$ that characterizes $(\alpha, \in)$. Consider any cardinal κ such that $(\kappa) \models \aleph_\varphi$. It follows that κ is uncountable and so $\kappa = \aleph_\beta$ for some ordinal $\beta > 0$ such that $(\beta, \in) \models \varphi$. Since φ characterizes α, $\alpha = \beta$ and so $\kappa = \aleph_\alpha$. Thus, $\aleph_\varphi$ characterizes $\aleph_\alpha$. This means that if $(\alpha, \in)$ is finitely characterizable then so is $\aleph_\alpha$. Now suppose that $(\alpha, \in)$ is infinitely characterizable. To show that $\aleph_\alpha$ is also infinitely characterizable it suffices to show that, for any nonzero $\kappa \neq \aleph_\alpha$, there is some sentence ψ of L_{2P} such that $(\aleph_\alpha) \models \psi$ and $(\kappa) \not\models \psi$. If κ is countable then, for any φ in the theory of $(\alpha, \in)$, $(\aleph_\alpha) \models \aleph_\varphi$ and $(\kappa) \not\models \aleph_\varphi$. If κ is uncountable then $\kappa = \aleph_\beta$ for some nonzero $\beta \neq \alpha$. Since $(\alpha, \in)$ is infinitely characterizable it follows that there is φ such that $(\alpha, \in) \models \varphi$ but $(\beta, \in) \not\models \varphi$, so $(\aleph_\alpha) \models \aleph_\varphi$ but $(\kappa) \not\models \aleph_\varphi$. Hence, $\aleph_\alpha$ is infinitely characterizable.

The construction of $\aleph_\varphi$, will use a formula $\textit{MYORD}(S, R)$ that, for any $\alpha > 0$, *characterizes* $(\alpha, \in)$ *in* $(\aleph_\alpha)$ in the sense that, for all $\chi \subseteq \aleph_\alpha$ and all ρ a binary relation on $\aleph_\alpha$, $(\aleph_\alpha) \models \textit{MYORD}(S, R)[\chi, \rho]$ iff $(\chi, \rho \restriction \chi)$ is isomorphic to $(\alpha, \in)$. This formula is constructed below.

The construction of $\aleph_\varphi$ also makes use of relativization of formulas to sets, and of substitution of variables for non-logical constants. Where φ is any formula, and S is any unary predicate variable, $\varphi[S]$ is the relativization of φ to S, as defined on page 150 of [8]. If φ is in $L_{2M(Q)}$ then so is $\varphi[S]$. If φ is a sentence of L_{2P}, then $\varphi[S]$ is a pure formula and $(A) \vDash \varphi[S][\chi]$ iff $(\chi) \vDash \varphi$. If φ is a sentence of $L_{2(Q)}$, then $(A, \rho) \vDash \varphi[S][\chi]$ iff $(\chi, \rho \restriction \chi) \vDash \varphi$. Given φ, a formula of $L_{2(Q)}$, and R, a binary relational variable, $\varphi[R]$ is the result of replacing every instance of Q in φ with R. $\varphi[S, R]$ is $\varphi[S][R]$. Where φ is a sentence of $L_{2(Q)}$, $(A) \vDash \varphi[S, R][\chi, \rho]$ iff $(\chi, \rho \restriction \chi) \vDash \varphi$.

These tools, combined with a formula to assert that the domain is uncountable, make it easy to construct $\aleph_\varphi$, as well as the formulas needed to prove Corollaries 2.3 and 2.4.

Before going further, it will be helpful to define a toolbox of formulas of L_{2P} to express simple set-theoretic notions. To begin, $\textsc{inj}(f)$ is given in 1, and $\mathfrak{A} \vDash \textsc{inj}(f)[h]$ iff h is one-to-one.

$$\forall xy(f(x) = f(y) \supset x = y) \tag{1}$$

$\textsc{subs}(S, S')$ and $\textsc{prosubs}(S, S')$ are given in 2 and 3, respectively. Note that $\mathfrak{A} \vDash \textsc{subs}(S, S')[\chi, \chi']$ iff $\chi \subseteq \chi'$ and $\mathfrak{A} \vDash \textsc{prosubs}(S, S')[\chi, \chi']$ iff $\chi \subset \chi'$.

$$\forall x(Sx \supset S'x) \tag{2}$$

$$\textsc{subs}(S, S') \wedge \neg\textsc{subs}(S', S) \tag{3}$$

Next, 4 gives $\textsc{le}(S, S')$, 5 gives $\textsc{lt}(S, S')$, and 6 gives $\textsc{eq}(S, S')$. Where $card(\chi)$ is the cardinality of χ, $\mathfrak{A} \vDash \textsc{le}(S, S')[\chi, \chi']$ iff $card(\chi) \leq card(\chi')$, $\mathfrak{A} \vDash \textsc{lt}(S, S')[\chi, \chi']$ iff $card(\chi) < card(\chi')$, and $\mathfrak{A} \vDash \textsc{eq}(S, S')[\chi, \chi']$ iff $card(\chi) = card(\chi')$.

$$\exists f(\textsc{inj}(f) \wedge \forall x(Sx \supset S'f(x))) \tag{4}$$

$$\textsc{le}(S, S') \wedge \neg\textsc{le}(S', S) \tag{5}$$

$$\textsc{le}(S, S') \wedge \textsc{le}(S', S) \tag{6}$$

The formula in 7 is $\textsc{dom}(S)$, and 8 is $\textsc{ld}(S)$.

$$\forall x Sx \tag{7}$$

$$\exists P(\textsc{dom}(P) \wedge \textsc{lt}(S, P)) \tag{8}$$

$\mathfrak{A} \vDash \textsc{dom}(S)[\chi]$ iff $\chi = A$ and $\mathfrak{A} \vDash \textsc{ld}(S)[\chi]$ iff $card(\chi) < card(A)$. Finally, 9 gives $\textsc{inf}(S)$ and 10 gives $\textsc{unc}$. $\mathfrak{A} \vDash \textsc{inf}(S)[\chi]$ iff χ is infinite, and $\mathfrak{A} \vDash \textsc{unc}$ iff $card(A) > \aleph_0$.

$$\exists P(\textit{\scriptsize PROSUBS}(P,S) \wedge \textit{\scriptsize EQ}(P,S)) \tag{9}$$

$$\exists S(\textit{\scriptsize INF}(S) \wedge \textit{\scriptsize LD}(S)) \tag{10}$$

To prove that $\textit{\scriptsize MYORD}(S,R)$ performs as advertised, it will be helpful to define, for any nonempty set A and any ρ a relation on A, a function $f_{\rho,A} : A \to \{\kappa \mid \kappa \leq card(A)\}$, the *cardinal indexing function on A induced by ρ.* This is the function such that, for all $a \in A$, $f_{\rho,A}(a) = card(\{b \mid (a,b) \in \rho\})$. Just as, on p. 103 of [8], binary relations are used to code information about families of sets, the above uses binary relations to code information about families of cardinals and about their structure. The argument proceeds by showing that various algebraic properties of this function can be expressed by second-order formulas, eventually showing that $(A) \vDash \textit{\scriptsize MYORD}(S,R)[\chi,\rho]$ iff there is ρ', a relation on A, such that $f_{\rho',A} \restriction \chi$ is an isomorphism between $(\chi, \rho \restriction \chi)$ and $(\{\kappa \geq \omega \mid \kappa \leq card(A)\}, \in)$.

The first formula to consider is used to talk about the image of an object under $f_{\rho,A}$. The pure formula 11 is called $\textit{\scriptsize INDX}(x,S,R)$, named mnemonically for "x indexes the cardinality of S with respect to R":

$$\exists P(\forall u(Pu \equiv Rxu) \wedge \textit{\scriptsize EQ}(S,P)) \tag{11}$$

It is easy to see that $\mathfrak{A} \vDash \textit{\scriptsize INDX}(x,S,R)[a,\chi,\rho]$ iff $f_{\rho,A}(a) = card(\chi)$.

The pure formula $\textit{\scriptsize ONETOONE}(S,R)$ is

$$\forall vw((Sv \wedge Sw \wedge \exists P'(\textit{\scriptsize INDX}(v,P',R) \wedge \textit{\scriptsize INDX}(w,P',R))) \supset v = w) \tag{12}$$

$\mathfrak{A} \vDash \textit{\scriptsize ONETOONE}(S,R)[\chi,\rho]$ iff $f_{\rho,A} \restriction \chi$ is one-to-one.

The pure formula in $\textit{\scriptsize INIMAGE}(S,S',R)$ is given by 13.

$$\exists v(S'v \wedge \textit{\scriptsize INDX}(v,S,R)) \tag{13}$$

$\mathfrak{A} \vDash \textit{\scriptsize INIMAGE}(S,S'R)[\chi,\chi'\rho]$ iff there is $a \in \chi'$ such that $f_{\rho,A}(a) = card(\chi)$, or, equivalently, iff χ is in the image of χ' under $f_{\rho,A}$.

Instead of one formula to say $f_{\rho,A}$ is surjective, it will be handy to have a general construction to say that the image of some set under $f_{\rho,A}$, is the set of cardinals defined by a given formula. Given a formula $\psi(S')$, $\textit{\scriptsize ONTO}^{\psi(S')}(S,R)$ is given by 14. If $\psi(S')$ is pure, then so is $\textit{\scriptsize ONTO}^{\psi(S')}(S,R)$.

$$\forall P(\exists P'(\textit{\scriptsize EQ}(P,P') \wedge \psi(P')) \equiv \textit{\scriptsize INIMAGE}(P,S,R)) \tag{14}$$

Lemma 2.1. *$\mathfrak{A} \vDash \textit{\scriptsize ONTO}^{\psi(S')}(S,R)[\chi,\rho]$ iff $f_{\rho,A}[\chi] = \{\lambda \mid$ there is $\chi' \subseteq A$ s.t. $card(\chi') = \lambda$ and $\mathfrak{A} \vDash \psi(S')[\chi']\}$.*

Proof. Suppose $\mathfrak{A} \vDash \textit{ONTO}^{\psi(S')}(S,R)[\chi,\rho]$. It follows that for all $\chi'' \subseteq A$, there is $\chi' \subseteq A$ such that $card(\chi') = card(\chi'')$ and $\mathfrak{A} \vDash \psi(S')[\chi']$ iff there is $a \in \chi$ such that $f_{\rho,A}(a) = card(\chi'')$. This in turn means that $f_{\rho,A}[\chi] = \{\lambda \mid$ there is $\chi' \subseteq A$ s.t. $card(\chi') = \lambda$ and $\mathfrak{A} \vDash \psi(S')[\chi']\}$. On the other hand, if $f_{\rho,A}[\chi] = \{\lambda \mid$ there is $\chi' \subseteq A$ s.t. $card(\chi') = \lambda$ and $\mathfrak{A} \vDash \psi(S')[\chi']\}$, then for all χ'' there is $\chi' \subseteq A$ such that $card(\chi') = card(\chi'')$ and $\mathfrak{A} \vDash \psi(S')[\chi']$ iff there is $a \in \chi$ such that $f_{\rho,A}(a) = card(\chi')$. This means that for all χ'', $\mathfrak{A} \vDash \textit{INIMAGE}(P',S,R)[\chi'',\chi,\rho]$ iff there is $\chi' \subseteq A$ such that $card(\chi') = card(\chi'')$ and $\mathfrak{A} \vDash \psi(S')[\chi']$. That is, $\mathfrak{A} \vDash \textit{ONTO}^{\psi(S')}(S,R)[\chi,\rho]$. ■

In particular, note that $\mathfrak{A} \vDash \textit{ONTO}^{INF(S') \wedge LD(S')}(S,R)[\chi,\rho]$ iff $f_{\rho,A}[\chi] = \{\lambda \mid \aleph_0 \leq \lambda < card(A)\}$.

The pure formula $\textit{HOM}(R',R,S)$ is given by 15.

$$\forall vw \forall PP' \left(\begin{array}{c} (Sv \wedge Sw \wedge \textit{INDX}(v,P,R) \wedge \textit{INDX}(w,P',R)) \\ \supset (R'vw \equiv \textit{LT}(P,P')) \end{array} \right) \tag{15}$$

$\mathfrak{A} \vDash \textit{HOM}(R',R,S)[\rho',\rho,\chi]$ iff $f_{\rho,A} \upharpoonright \chi$ is a homomorphism from $(\chi, \rho' \upharpoonright \chi)$ to $(\{\lambda \mid \lambda \leq card(A)\}, \in)$.

The above can be combined to build the pure formula $\textit{MYORD}(S,R)$ that, for $\alpha > 0$, characterizes $(\alpha,\in)$ in $(\aleph_\alpha)$. This pure formula is

$$\exists R'(\textit{ONTO}^{INF(S') \wedge LD(S')}(S,R') \wedge \textit{ONETOONE}(S,R') \wedge \textit{HOM}(R,R',S)) \tag{16}$$

Lemma 2.2. *For any $\alpha > 0$, $\textit{MYORD}(R,S)$ characterizes $(\alpha,\in)$ in $(\aleph_\alpha)$.*

Proof. Observations above about the formulas making up $\textit{MYORD}(S,R)$ make it easy to see that $(\aleph_\alpha) \vDash \textit{MYORD}(S,R)[\chi,\rho]$ iff there is ρ' such that:

1. $f_{\rho',\aleph_\alpha} \upharpoonright \chi$ is one-to-one
2. $f_{\rho',\aleph_\alpha}[\chi] = \{\lambda \mid \aleph_0 \leq \lambda < \aleph_\alpha\}$
3. $f_{\rho',\aleph_\alpha} \upharpoonright \chi$ is a homomorphism from $(\chi, \rho \upharpoonright \chi)$ to $(\{\lambda \mid \lambda \leq \aleph_\alpha\}, \in)$

This means that $(\aleph_\alpha) \vDash \textit{MYORD}(S,R)[\chi,\rho]$ iff there is ρ' such that $f_{\rho',\aleph_\alpha} \upharpoonright \chi$ is an isomorphism from $(\chi, \rho \upharpoonright \chi)$ to $(\{\lambda \mid \aleph_0 \leq \lambda < \aleph_\alpha\}, \in)$. It follows from the definition of $\aleph_\alpha$ that $(\{\lambda \mid \aleph_0 \leq \lambda < \aleph_\alpha\}, \in)$ is isomorphic to $(\alpha,\in)$. Thus, if $(\aleph_\alpha) \vDash \textit{MYORD}(S,R)[\chi,\rho]$ it follows immediately that $(\chi, \rho \upharpoonright \chi)$ is isomorphic to $(\{\lambda \mid \aleph_0 \leq \lambda < \aleph_\alpha\}, \in)$ and so to $(\alpha,\in)$. On the other hand, given $\chi \subseteq \aleph_\alpha$ and $\rho \subseteq \aleph_\alpha \times \aleph_\alpha$ such that $(\chi, \rho \upharpoonright \chi)$ is isomorphic to $(\alpha,\in)$, it follows that $(\chi, \rho \upharpoonright \chi)$ is also isomorphic to $(\{\lambda \mid \aleph_0 \leq \lambda < \aleph_\alpha\}, \in)$. Let f be an isomorphism from $(\chi, \rho \upharpoonright \chi)$ to $(\{\lambda \mid \aleph_0 \leq \lambda < \aleph_\alpha\}, \in)$. Let

$\rho' = \{(\beta, \gamma) \mid \beta \in \chi, \gamma \in \aleph_\alpha, \gamma \in f(\beta)\}$. $f_{\rho',\aleph_\alpha}(\beta) = card(f(\beta))$ for all $\beta \in \chi$, so $f_{\rho',\aleph_\alpha}$ is an isomorphism from $(\chi, \rho \restriction \chi)$ to $(\{\lambda \mid \aleph_0 \leq \lambda < \aleph_\alpha\}, \in)$, which is to say that there is ρ', a binary relation of $\aleph_\alpha$, such that $f_{\rho',\aleph_\alpha} \restriction \chi$ is one-to-one, is a homomorphism from $(\chi, \rho \restriction \chi)$ to $(\{\lambda \mid \lambda \leq \aleph_\alpha\}, \in)$, and is onto $\{\lambda \mid \aleph_0 \leq \lambda < \aleph_\alpha\}$, but as noted earlier these are exactly the truth-conditions of $\textit{MYORD}(S, R)$, so $\aleph_\alpha \models \textit{MYORD}(S, R)[\chi, \rho]$. ■

If $card(A) = \aleph_\alpha$ then, since isomorphic structures are second-order equivalent, $\mathfrak{A} \models \textit{MYORD}(S, R)[\chi, \rho]$ iff $(\chi, \rho \restriction \chi)$ is isomorphic to $(\alpha, \in)$.

$\textit{MYORD}(S, R)$ allows us to map the second-order theory of any $(\alpha, \in)$ onto the second-order theory of $(\aleph_\alpha)$, by associating with each sentence φ in the theory of $(\alpha, \in)$ a pure sentence $\aleph_\varphi$ given by 17.

$$\textit{UNC} \wedge \exists S \exists R(\textit{MYORD}(S, R) \wedge \varphi[S, R]) \tag{17}$$

This formula is pure because there is only one non-logical constant in the theory of $(\alpha, \in)$ and it is systematically replaced by a relational variable.

THEOREM 2.1. *For any* $\alpha > 0$, $(\alpha, \in) \models \varphi$ *iff* $(\aleph_\alpha) \models \aleph_\varphi$.

PROOF. $(\aleph_\alpha) \models \aleph_\varphi$ iff $\aleph_\alpha$ is uncountable and there are $\chi \subseteq \aleph_\alpha$, $\rho \subseteq \aleph_\alpha \times \aleph_\alpha$ such that $(\chi, \rho \restriction \chi)$ is isomorphic to $(\alpha, \in)$ and $(\chi, \rho \restriction \chi) \models \varphi$. But if this is the case then by the indistinguishability of isomorphic structures $(\alpha, \in) \models \varphi$. On the other hand, for $\alpha > 0$ such that $(\alpha, \in) \models \varphi$, it follows that $\aleph_\alpha$ is uncountable so $(\aleph_\alpha) \models \textit{UNC}$, and since $card(\alpha) \leq \aleph_\alpha$ and $\aleph_\alpha$, being a cardinal, is also an ordinal, it follows that $\alpha \subseteq \aleph_\alpha$, likewise it follows that $\in \restriction \alpha$ is a relation on $\aleph_\alpha$, and by Lemma 2.2 $(\aleph_\alpha) \models \textit{MYORD}(S, R)[\alpha, \in \restriction \alpha]$, so $(\aleph_\alpha) \models \aleph_\varphi$. ■

Thus, for $\alpha > 0$ and $\mathfrak{A}$ such that $card(A) = \aleph_\alpha$, $(\alpha, \in) \models \varphi$ iff $\mathfrak{A} \models \aleph_\varphi$. Arguments from the beginning of the section now give us the following:

COROLLARY 2.1. *For any* $\alpha > 0$, *if* $(\alpha, \in)$ *is finitely characterizable then so is* $\aleph_\alpha$.

COROLLARY 2.2. *For any* $\alpha > 0$, *if* $(\alpha, \in)$ *is infinitely characterizable then so is* $\aleph_\alpha$.

In [3] (p. 84) the result corresponding to Corollary 2.1 is a special case of a more general result. Related results appear below. Where ψ is any sentence of L_{2P}, $\textit{BIJSPEC}(\psi)(S, R, R')$ is given by 18.

$$\textit{ONTO}^{\psi[S'] \wedge \exists x S' x}(S, R') \wedge \textit{ONETOONE}(S, R') \wedge \textit{HOM}(R, R', S) \tag{18}$$

Arguments like those above can be used to show that $\mathfrak{A} \models \textsc{bijspec}(\psi)(S, R, R')[\chi, \rho, \rho']$ iff $f_{\rho',A} \restriction \chi$ is an isomorphism from $(\chi, \rho \restriction \chi)$ to $(\{\kappa > 0 \mid (\kappa) \models \psi\}, \in)$.

Next, $\varphi_{inSPEC}(\psi)$ is the pure sentence given by 19.

$$\psi \wedge \exists S \exists R R'(\textsc{bijspec}(\psi)(s, R, R') \wedge \varphi[S, R]) \tag{19}$$

THEOREM 2.2. *If the spectrum of ψ contains only infinite cardinals, $(\kappa) \models \varphi_{inSPEC}(\psi)$ iff there is some $\alpha > 0$ such that $(\alpha, \in) \models \varphi$ and κ is the α^{th} cardinal in the spectrum of ψ (in the sense that $(\kappa) \models \psi$ and $(\{\lambda \mid \lambda < \kappa$ and $(\lambda) \models \psi\}, \in)$ is isomorphic to $(\alpha, \in)$).*

This result can be proven using essentially the same arguments that were used to prove Lemma 2.2 and Theorem 2.1 above.

Building on Theorem 2.2 in the same manner as on Theorem 2.1 yields:

COROLLARY 2.3. *For any $\alpha > 0$, ψ a pure sentence, if the spectrum of ψ contains only infinite cardinals and $(\alpha, \in)$ is finitely characterizable, then so is the α^{th} cardinal that satisfies ψ.*

COROLLARY 2.4. *For any $\alpha > 0$, ψ a pure sentence, if the spectrum of ψ contains only infinite cardinals and $(\alpha, \in)$ is infinitely characterizable, then so is the α^{th} cardinal that satisfies ψ.*

3. The Fraenkel-Carnap Question for Strict Well-Orders

This section investigates one way of describing the set of finitely characterizable ordinals: $(\alpha, \in)$ is finitely characterizable iff its second-order theory is finitely axiomatizable. That is, the class of strict well-orders has the second-order Fraenkel-Carnap property, as defined in Section 1. The approach used here is similar to the one used in [12], but it has some differences, among them that it does not lead naturally to a proof of the quasi Fraenkel-Carnap property (defined in [12] on page 286), but does provide a proof that the strict well-orders have the monadic second-order Fraenkel-Carnap property.

To begin, note that there is an axiom of strict well-ordering in $L_{2M(Q)}$. Let 20 be called $\textsc{trans}$, 21 be called $\textsc{sto}$, and 22 be called $\textsc{swo}$:

$$\forall xyz((Qxy \wedge Qyz) \supset Qxz) \tag{20}$$

$$\textsc{trans} \wedge \neg\exists xy(Qxy \wedge Qyx) \wedge \forall zw(Qwz \vee Qzw \vee w = z) \tag{21}$$

$$\textsc{sto} \wedge \forall P(\exists v Pv \supset \exists x(Px \wedge \forall y(Py \supset (x = y \vee Qxy)))) \tag{22}$$

Observe that $(A, \rho) \vDash \textit{TRANS}$ iff ρ is transitive on A, 21 is an axiom of strict total-ordering, and 22 is an axiom of strict well-ordering.

Next, $\textit{INSEG}(S)$ is the monadic formula

$$\exists x \forall y (Sy \equiv Qyx) \tag{23}$$

Given (A, ρ), a strict well-order, $(A, \rho) \vDash \textit{INSEG}(S)[\chi]$ iff χ is a strict initial segment of (A, ρ).

The definitions above make it possible, for any second-order sentence φ, to construct a sentence $\varphi^{\Downarrow}$ that is satisfied only by the least strict well-order that satisfies φ. Given φ, $\varphi^{\Downarrow}$ is

$$\textit{SWO} \wedge \varphi \wedge \forall S((\textit{INSEG}(S) \wedge \exists x Sx) \supset \neg\varphi[S]) \tag{24}$$

Note that if φ is monadic then so is $\varphi^{\Downarrow}$.

Theorem 3.1. *If there is at least one strict well-order* (A, ρ') *such that that* $(A, \rho') \vDash \varphi$*, then* $(B, \rho) \vDash \varphi^{\Downarrow}$ *iff* (B, ρ) *is isomorphic to* $(\alpha, \in)$*, for the least ordinal* α *such that* $(\alpha, \in) \vDash \varphi$*.*

Proof. Suppose (B, ρ) is isomorphic to $(\alpha, \in)$, for the least ordinal α such that $(\alpha, \in) \vDash \varphi$. It follows immediately that $(B, \rho) \vDash \varphi$ and that $(B, \rho) \vDash \textit{SWO}$. It also follows that $(B, \rho) \vDash \forall S((\textit{INSEG}(S) \wedge \exists x Sx) \supset \neg\varphi[S])$, for, if $(B, \rho) \nvDash \forall S((\textit{INSEG}(S) \wedge \exists x Sx) \supset \neg\varphi[S])$ then there is χ a nonempty strict initial segment of (B, ρ) such that $(B, \rho) \vDash \varphi[S][\chi]$, which is to say $(\chi, \rho \restriction \chi) \vDash \varphi$. Since χ is a strict initial segment, it follows that $(\chi, \rho \restriction \chi)$ is isomorphic to $(\beta, \in)$ for some $0 < \beta < \alpha$. Thus there is $\beta < \alpha$ such that $(\beta, \in) \vDash \varphi$, contradicting the hypothesis that α is minimal. On the other hand, suppose $(B, \rho) \vDash \varphi^{\Downarrow}$. It follows that $(B, \rho) \vDash \varphi$ and (B, ρ) is a strict well-order. Being a strict well-order, (B, ρ) is isomorphic to $(\gamma, \in)$ for some ordinal $\gamma > 0$. α, as defined above, cannot be greater than γ, but supposed that it is less than γ. Since α will then be a strict initial segment of $(\gamma, \in)$, it follows that $(\alpha, \in)$ is isomorphic to a nonempty strict initial segment of (B, ρ). Let $\chi \subset B$ be the image of α under this isomorphism. It follows that $(B, \rho) \vDash \textit{INSEG}(S)[\chi]$, $(B, \rho) \vDash \exists x Sx[\chi]$, and $(B, \rho) \vDash \varphi[S][\chi]$. This implies that $(B, \rho) \nvDash \forall S((\textit{INSEG}(S) \wedge \exists x Sx) \supset \neg\varphi[S])$, which contradicts the assumption that $(B, \rho) \vDash \varphi^{\Downarrow}$. Thus, $\gamma = \alpha$, so (B, ρ) is isomorphic to $(\alpha, \in)$. ■

Given the above result, it is easy to show that the class of strict well-orders has the Fraenkel-Carnap property:

Corollary 3.1. *A strict well-order (A, ρ) is finitely characterizable iff the second-order theory of (A, ρ) is finitely axiomatizable.*

Proof. If (A, ρ) is finitely characterizable, let φ be the sentence that characterizes (A, ρ). For every sentence ψ, $\varphi \vDash \psi$ iff ψ is satisfied by every model of φ, which is to say iff $(A, \rho) \vDash \psi$, so φ entails exactly those sentences that are in the theory of (A, ρ). On the other hand, suppose that the second-order theory of (A, ρ) is finitely axiomatizable, and let φ be a sentence in the second-order theory of (A, ρ) that entails all other sentences in that theory; clearly $(A, \rho) \vDash \varphi$ and, for ever sentence ψ of $L_{2(Q)}$, either $\varphi \vDash \psi$ (if ψ is in the theory of (A, ρ)) or $\varphi \vDash \neg\psi$ (if ψ is not in the theory of (A, ρ)). Now consider $\varphi^{\Downarrow}$. It follows from 3.1 that $\varphi^{\Downarrow}$ is satisfiable and that all models of $\varphi^{\Downarrow}$ are isomorphic. By its construction, $\varphi^{\Downarrow}$ is satisfied by some model of φ, so $\varphi \nvDash \neg\varphi^{\Downarrow}$, so it must be the case that $\varphi \vDash \varphi^{\Downarrow}$. Thus, since all models of $\varphi^{\Downarrow}$ are isomorphic, all models of φ are isomorphic as well, so, since $(A, \rho) \vDash \varphi$, φ characterizes (A, ρ) and (A, ρ) is finitely characterizable. ■

Everything done above goes through virtually unchanged if attention is restricted to sentence of $L_{2M(Q)}$, so a strict well-order (A, ρ) is finitely characterized by a monadic formula iff there is a monadic formula φ such that $(A, \rho) \vDash \varphi$ and, for all sentences ψ of $L_{2M(Q)}$, either $\varphi \vDash \psi$ or $\varphi \vDash \neg\psi$. Likewise, $\varphi^{\Downarrow}$ could be constructed for any φ in any of a number of stronger logics, including third-order logic. The present argument can also be easily amended to work for reflexive well-orders instead of strict well-orders.

4. Characterizability and Cardinal Arithmetic

This section explores some sufficient conditions for finite and infinite cardinal characterizability, in the form of "closure" properties of the finitely and infinitely characterizable cardinals under well-known operations. Most of the results for finite characterizability are stated in [8], and related results appear in [3] and [4]. In what follows, two of the results about finite characterizability are paralleled by analogous results about infinite characterizability.

To begin, observe that it doesn't matter whether one talks about the characterizability of a cardinal number as a set or as an order structure, because of the pure formula *initial*, given in 25 below.

$$\textit{swo} \wedge \forall S(\textit{inseg}(S) \supset \textit{ld}(S)) \tag{25}$$

Clearly $(\alpha, \in) \vDash$ *initial* iff α is the least ordinal of cardinality $card(\alpha)$ (that is, if α is a cardinal). Furthermore, for $\alpha > 0$, if $(\alpha, \in) \vDash \varphi$ then

$(card(\alpha)) \vDash \exists R\varphi[R]$. So, in particular, if a cardinal κ is characterized by a pure sentence φ, then $(\kappa, \in)$ is characterized by the sentence $\varphi \wedge \textsc{initial}$, and if $(\kappa, \in)$ is characterized by a sentence ψ, then κ is characterized by $\exists R\psi[R]$. This leads to Proposition 4.1, similar to a result in [4].

Proposition 4.1. *A cardinal $\kappa > 0$ is finitely characterizable iff $(\kappa, \in)$ is finitely characterizable.*

If κ is infinitely characterizable, then for every ordinal $\alpha > 0$ such that $\alpha \neq \kappa$, either $\kappa \neq card(\alpha)$ or else α is not a cardinal. If the former, then since κ is infinitely characterizable there is a pure sentence φ such that $(\kappa) \vDash \varphi$ and $(\alpha) \nvDash \varphi$, and so $(\kappa, \in) \vDash \varphi$ and $(\alpha, \in) \nvDash \varphi$. If the latter, then $(\kappa, \in) \vDash \textsc{initial}$ and $(\alpha, \in) \nvDash \textsc{initial}$. Thus, $(\kappa, \in)$ is infinitely characterizable. Going the other way, if $(\kappa, \in)$ is infinitely characterizable, then for every cardinal $\lambda \neq \kappa$, there is φ such that $(\kappa, \in) \vDash \varphi$ and $(\lambda, \in) \nvDash \varphi$, so $(\kappa) \vDash \exists R(\varphi[R] \wedge \textsc{swo}[R] \wedge \textsc{initial}[R])$ while $(\lambda) \nvDash \exists R(\varphi[R] \wedge \textsc{swo}[R] \wedge \textsc{initial}[R])$. Thus:

Proposition 4.2. *A cardinal $\kappa > 0$ is infinitely characterizable iff $(\kappa, \in)$ is infinitely characterizable.*

The above combine Corollaries 2.1 and 2.2 to yield these closure results:

Proposition 4.3. *If κ is finitely characterizable then so is $\aleph_\kappa$.*

Proposition 4.4. *If κ is infinitely characterizable then so is $\aleph_\kappa$.*

The other closure results are for familiar operations from cardinal arithmetic. Both are shown using a formula to characterize a smaller cardinal inside of a larger one. First, the pure formula $\textsc{mycpred}(S)$ is

$$\textsc{ld}(S) \wedge \forall P(\textsc{ld}(P) \supset \textsc{le}(P, S)) \tag{26}$$

$\mathfrak{A} \vDash \textsc{mycpred}(S)[\chi]$ iff $card(A)$ is the cardinal successor of $card(\chi)$. Now, given a pure formula φ, the pure formula φ^+ is

$$\exists S(\textsc{mycpred}(S) \wedge \varphi[S] \wedge \exists x Sx) \tag{27}$$

For $\kappa > 0$, if $(\kappa) \vDash \varphi$, it follows that $(\kappa^+) \vDash \varphi[S][\kappa]$, $(\kappa^+) \vDash \textsc{mycpred}(S)[\kappa]$, and $(\kappa^+) \vDash \exists x Sx[\kappa]$, so $(\kappa^+) \vDash \varphi^+$. On the other hand, if $(\kappa^+) \vDash \varphi^+$, then there is $\chi \subseteq \kappa^+$ such that $card(\chi) = \kappa$, and $(\kappa^+) \vDash \varphi[S][\chi]$, hence, $(\chi) \vDash \varphi$, so $(\kappa) \vDash \varphi$. Proposition 4.5 is immediate:

Proposition 4.5. *For any cardinal $\kappa > 0$, $(\kappa) \vDash \varphi$ iff $(\kappa^+) \vDash \varphi^+$.*

If $\kappa > 0$ is finitely characterizable, characterized by φ, then it is easy to see that κ^+ is characterized by φ^+. This observation leads to the following result, which has been noted in [8] and resembles a result in [3]:

Proposition 4.6. *If κ is finitely characterizable then so is κ^+.*

If $\kappa > 0$ is infinitely characterizable, then consider any $\lambda > 0$ such that $\lambda \neq \kappa^+$. If $\lambda = 1$ or λ is not a successor cardinal then, where ψ is any logically true pure sentence, $(\kappa^+) \vDash \psi^+$ and $(\lambda) \nvDash \psi^+$, but if λ is a successor cardinal and $\lambda > 1$ then let μ be the cardinal such that $\lambda = \mu^+$. Since κ is infinitely characterizable there is φ such that $(\kappa) \vDash \varphi$ and $(\mu) \nvDash \varphi$, so it follows that $(\kappa^+) \vDash \varphi^+$ and $(\lambda) \nvDash \varphi^+$. Thus, no $\lambda \neq \kappa^+$ satisfies the same pure sentences as κ^+, yielding the following:

Proposition 4.7. *If κ is infinitely characterizable then so is κ^+.*

Reproduced below is an argument, based on one from [8], that if κ is finitely characterizable then 2^κ is as well. Unfortunately, since the function involved is not known to be one-to-one, the previous approach cannot extend this result to cover infinite characterizability, so there remains a point of asymmetry. As in [8], the formulas used can be regarded as statements about the set-indexing function induced by a relation. To begin, the pure formula $\textit{\small DISTNEIGHBORHOODS}(x, y, R)$ is

$$\exists w((Rxw \wedge \neg Ryw) \vee (Ryw \wedge \neg Rxw)) \tag{28}$$

$\mathfrak{A} \vDash \textit{\small DISTNEIGHBORHOODS}(x, y, R)[a, b, \rho]$ iff $\{c \in A \mid (a, c) \in \rho\} \neq \{c \in A \mid (b, c) \in \rho\}$. Next, the pure formula $\textit{\small ALLNBHDIST}(R)$ is

$$\forall yz(y \neq z \supset \textit{\small DISTNEIGHBORHOODS}(y, z, R)) \tag{29}$$

Observe that $\mathfrak{A} \vDash \textit{\small ALLNBHDIST}(R)[\rho]$ iff $\{c \in A \mid (a, c) \in \rho\} \neq \{c \in A \mid (b, c) \in \rho\}$, for all distinct $a, b \in A$. The pure formula $\textit{\small AM2TOCARD}(S)$ is

$$\exists R(\forall P(\textit{\small SUBS}(P, S) \equiv \exists w \forall x(Rwx \equiv Px)) \wedge \textit{\small ALLNBHDIST}(R)) \tag{30}$$

Note that $\mathfrak{A} \vDash \textit{\small AM2TOCARD}(S)[\chi]$ iff $card(A) = 2^{card(\chi)}$.

Given φ a pure sentence, the pure sentence called 2^φ is

$$\exists S(\textit{\small AM2TOCARD}(S) \wedge \varphi[S]) \tag{31}$$

From the remarks on $\textit{\small AM2TOCARD}(S)$, the following is immediate:

Proposition 4.8. *For $\kappa > 0$, if $(\kappa) \vDash \varphi$ then $(2^\kappa) \vDash 2^\varphi$.*

From here, it is easy to see that:

Proposition 4.9. *If κ is finitely characterizable, then so is 2^κ.*

5. Characterizability and Ordinal Arithmetic

Turning to ordinal arithmetic, it is possible to apply the approach from Section 4 to a wider variety of operations. For binary operations, however, these techniques have at best partial success with infinite characterizability.

A few more formulas will be useful. 32 gives $\textit{\scriptsize INRANGE}(x, f)$ and 33 gives $\textit{\scriptsize RANGE}(S, f)$. Note that $\mathfrak{A} \vDash \textit{\scriptsize INRANGE}(x, f)[a, h]$ iff a is in the range of h, and that $\mathfrak{A} \vDash \textit{\scriptsize RANGE}(S, f)[\chi, h]$ iff $\chi = h[A]$.

$$\exists y(f(y) = x) \tag{32}$$

$$\forall x(Sx \equiv \textit{\scriptsize INRANGE}(x, f)) \tag{33}$$

The first operation considered will be ordinal successor. Begin with the monadic formula $\textit{\scriptsize MYOPRED}(S)$:

$$\textit{\scriptsize INSEG}(S) \wedge \forall xy((\neg Sx \wedge \neg Sy) \supset x = y) \tag{34}$$

For $\alpha > 0$, $(\alpha, \in) \vDash \textit{\scriptsize MYOPRED}(S)[\chi]$ iff α is a successor ordinal and $\chi = \alpha - 1$.

Next, given a sentence φ, $\varphi + 1$ is

$$\textit{\scriptsize SWO} \wedge \exists S(\textit{\scriptsize MYOPRED}(S) \wedge \varphi[S] \wedge \exists x Sx) \tag{35}$$

Note that if φ is monadic then so is $\varphi + 1$, and that $(A, \rho) \vDash \varphi + 1$ iff (A, ρ) is isomorphic to $(\alpha + 1, \in)$ for some $\alpha > 0$ and $(\alpha, \in) \vDash \varphi$. Thus, if $(\alpha, \in)$ is characterized by φ, $(\alpha + 1, \in)$ is characterized by $\varphi + 1$, leading to the following:

Proposition 5.1. *For any ordinal $\alpha > 0$, if $(\alpha, \in)$ is finitely characterizable, then so is $(\alpha + 1, \in)$.*

Now consider $\alpha > 0$ such that $(\alpha, \in)$ is infinitely characterizable, and $\beta > 0$ such that $\beta \neq \alpha + 1$. If $\beta = 1$ or β is not a successor ordinal, then where ψ is any logically true sentence, $(\alpha + 1, \in) \vDash \psi + 1$ and $(\beta, \in) \nvDash \psi + 1$. On the other hand, if $\beta > 1$ and β is a successor ordinal, then let $\gamma > 0$ be such that $\beta = \gamma + 1$. Since $(\alpha, \in)$ is infinitely characterizable, it follows that there is φ such that $(\alpha, \in) \vDash \varphi$ but $(\gamma, \in) \nvDash \varphi$, so $(\alpha + 1, \in) \vDash \varphi + 1$, but $(\beta, \in) \nvDash \varphi + 1$, thus:

Proposition 5.2. *For any ordinal $\alpha > 0$, if $(\alpha, \in)$ is infinitely characterizable, then so is $(\alpha + 1, \in)$.*

The proofs of 5.1 and 5.2 only require quantification over set and individual variables, so analogous results hold for characterizability in $L_{2M(Q)}$.

Below it is shown that the finitely and infinitely characterizable ordinals are closed under taking ordinal predecessors. Because an ordinal is not a subsystem of its predecessor, the formulas involved are somewhat more convoluted, and do not yield an analogous result in the monadic case.

To begin, $\textsc{embedsme}(f, R)$ is given by 36.

$$\textsc{inj}(f) \wedge \forall xy(Qxy \equiv Rf(x)f(y)) \tag{36}$$

$(A, \rho) \vDash \textsc{embedsme}(f, R)[h, \rho']$ iff (A, ρ) is isomorphic to $(h[A], \rho' \restriction h[A])$.

One can get a copy of the predecessor of a strict well-order by taking the subset containing the appropriate initial segment, but this won't work for the successor, since no subset of the domain has the necessary structure. Thus, a relational variable is required. $\textsc{myosucc}(R)$ is given by 37.

$$\exists f(\textsc{embedsme}(f, R) \wedge \textsc{swo}[R] \wedge \exists S(\textsc{range}(S, f) \wedge \textsc{myopred}(S)[R])) \tag{37}$$

Facts noted above imply that, if (A, ρ) is a strict well-order, then $(A, \rho) \vDash \textsc{myosucc}(R)[\rho']$ iff there is α such that (A, ρ) is isomorphic to $(\alpha, \in)$ such that (A, ρ') is isomorphic to $(\alpha + 1, \in)$. Further, if A is infinite and (A, ρ) is a strict well-order, then, since $\alpha + 1$ is of the same cardinality as α for any infinite α, it follows that there is ρ' such that $(A, \rho) \vDash \textsc{myosucc}(R)[\rho']$.

For any sentence φ, $\varphi - 1$ is given by 38. $(A, \rho) \vDash \varphi - 1$ iff there is $\alpha \geq \omega$ such that (A, ρ) is isomorphic to $(\alpha, \in)$ and $(\alpha + 1, \in) \vDash \varphi$.

$$\textsc{swo} \wedge (\exists S \textsc{inf}(S)) \wedge \exists R(\textsc{myosucc}(R) \wedge \varphi[R]) \tag{38}$$

Consider $\alpha \geq \omega$ such that $(\alpha+1, \in)$ is finitely characterizable, characterized by φ. It follows from the above that $(A, \rho) \vDash \varphi - 1$ iff (A, ρ) is isomorphic to $(\alpha, \in)$. In the case where $\omega > \alpha > 0$, both $(\alpha, \in)$ and $(\alpha + 1, \in)$ are finite structures and so can be finitely characterized, thus:

Proposition 5.3. *For any* $\alpha > 0$, *if* $(\alpha + 1, \in)$ *is finitely characterizable, then so is* $(\alpha, \in)$.

Now consider $\alpha \geq \omega$ and $\beta > 0$ such that $(\alpha + 1, \in)$ is infinitely characterizable and $\beta \neq \alpha$. Since $(\alpha + 1, \in)$ is infinitely characterizable, there is φ such that $(\alpha + 1, \in) \vDash \varphi$ and $(\beta + 1, \in) \nvDash \varphi$, thus $(\alpha, \in) \vDash \varphi - 1$ and $(\beta, \in) \nvDash \varphi - 1$. Of course, when $\omega > \alpha > 0$, $(\alpha, \in)$ and $(\alpha + 1, \in)$ are both finitely characterizable and so both infinitely characterizable. This leads to the following closure result:

Proposition 5.4. *For any $\alpha > 0$, if $(\alpha + 1, \in)$ is infinitely characterizable, then so is $(\alpha, \in)$.*

Next, ordinal addition and multiplication are considered, leading to closure results for the finitely characterizable ordinals, and somewhat weaker results for the infinitely characterizable ordinals.

To begin, consider the formula $\textsc{compl}(S, S')$, given by 39.

$$\forall x((Sx \vee S'x) \wedge \neg(Sx \wedge S'x)) \tag{39}$$

Observe that $\mathfrak{A} \vDash \textsc{compl}(S, S')[\chi, \chi']$ iff χ' is the complement of χ with respect to A.

Now, given sentences φ and ψ, $\varphi + \psi$ is

$$\textsc{swo} \wedge \exists SS'(\textsc{inseg}(S) \wedge \exists x Sx \wedge \textsc{compl}(S', S) \wedge \varphi[S] \wedge \psi[S']) \tag{40}$$

If φ and ψ are monadic, then so is $\varphi + \psi$.

For any φ and ψ, all models of $\varphi + \psi$ are strict well-orders, so assume, without loss of generality, that all models are ordinals. Let $\gamma = \alpha + \beta$, where $\alpha, \beta > 0$, $(\alpha, \in) \vDash \varphi$, and $(\beta, \in) \vDash \psi$. By the definition of ordinal addition, $\alpha \subseteq \gamma$ and, where χ is the complement of α in γ, $(\beta, \in)$ is isomorphic to $(\chi, \in)$. Thus, $(\gamma, \in) \vDash \varphi[S][\alpha]$ and $(\gamma, \in) \vDash \psi[S'][\chi]$, so $(\gamma, \in) \vDash \varphi + \psi$. Now consider any γ such that $(\gamma, \in) \vDash \varphi + \psi$. There must be α, a nonempty strict initial segment of $(\gamma, \in)$, such that $(\gamma, \in) \vDash \varphi[S][\alpha]$ and, where χ is the complement of α in γ, $(\gamma, \in) \vDash \psi[S'][\chi]$, so $(\alpha, \in) \vDash \varphi$ and $(\chi, \in) \vDash \psi$. Since χ is a nonempty subset of an ordinal, $(\chi, \in)$ is isomorphic to $(\beta, \in)$ for some $\beta > 0$, so $(\beta, \in) \vDash \psi$, which is to say $\gamma = \alpha + \beta$, $(\alpha, \in) \vDash \varphi$, and $(\beta, \in) \vDash \psi$, proving Lemma 5.1:

Lemma 5.1. *$(A, \rho) \vDash \varphi + \psi$ iff there are $\alpha, \beta > 0$ such that (A, ρ) is isomorphic to $(\alpha + \beta, \in)$, $(\alpha, \in) \vDash \varphi$, and $(\beta, \in) \vDash \psi$.*

The construction of such a formula for ordinal multiplication is somewhat more involved, and some tools for talking about binary functions will be required. In everything that follows, p is a binary functional variable of the object-language, and ξ is a metalanguage variable over binary functions.

The following two formulas are used to restrict attention to bijective binary functions. The first, given in 41, is called $\textsc{bininj}(p, S, S')$:

$$\forall wxyz((Sw \wedge Sx \wedge S'y \wedge S'z) \supset ((p(w, y) = p(x, z)) \equiv (w = z \wedge y = z))) \tag{41}$$

A formula called $\textsc{binsurj}(p, S, S', S'')$ is given in 42:

$$\forall xy((Sx \wedge S'y) \supset S''p(x, y)) \wedge \forall z(S''z \supset \exists vw(Sv \wedge S'w \wedge z = p(v, w))) \tag{42}$$

Note that $\mathfrak{A} \vDash \textsc{bininj}(p, S, S')[\xi, \chi, \chi']$ iff $\xi \restriction (\chi \times \chi')$ is one-to-one, and that $\mathfrak{A} \vDash \textsc{binsurj}(p, S, S', S'')[\xi, \chi, \chi', \chi'']$ iff ξ is surjective from $\chi \times \chi'$ onto χ''.

Next, 43 defines $\textsc{pairs}(p, S, S', S'')$. $\mathfrak{A} \vDash \textsc{pairs}(p, S, S', S'')[\xi, \chi, \chi', \chi'']$ iff $\xi \restriction (\chi \times \chi')$ is a bijection between χ' and χ''.

$$\textsc{bininj}(p, S, S') \wedge \textsc{binsurj}(p, S, S', S'') \tag{43}$$

$\textsc{pairs}(p, S, S', S'')$ makes it possible to talk about Cartesian products. To talk about products of strict well-orders, it is necessary to talk about the behavior of orderings on these products. Thus, $\textsc{lexord}(p, S, S')$ is

$$\forall wxyz \left(\begin{array}{c} (Sw \wedge Sy \wedge S'x \wedge S'z) \supset \\ (Qp(w, x)p(y, z) \equiv (Qxz \vee (x = z \wedge Qwy))) \end{array} \right) \tag{44}$$

Consider (A, ρ), a strict well-order, χ and χ' nonempty subsets of A, and ξ a binary function on A, such that $(A, \rho) \vDash \textsc{lexord}(p, S, S')[\xi, \chi, \chi']$. For $a, b \in \chi$ and $a', b' \in \chi'$, it follows from observations above that $(\xi(a, a'), \xi(b, b')) \in \rho$ iff either $(a', b') \in \rho$ or $a' = b'$ and $(a, b) \in \rho$, which is equivalent to the assertion that $(\xi(a, a'), \xi(b, b')) \in \rho$ iff $((a, a'), (b, b')) \in \varsigma$, where ς is the ordering relation of the order product $(\chi, \rho \restriction \chi) \otimes (\chi', \rho \restriction \chi')$. On the other hand, suppose that ξ has the property that if $a, b \in \chi$ and $a', b' \in \chi'$ then $(\xi(a, a'), \xi(b, b')) \in \rho$ iff $((a, a'), (b, b')) \in \varsigma$; this is equivalent to saying that $(\xi(a, a'), \xi(b, b')) \in \rho$ iff $(a', b') \in \rho$ or $a' = b'$ and $(a, b) \in \rho$, which in turn means that for all $a, b \in \chi$ and $a', b' \in \chi'$, $(A, \rho) \vDash (Qp(w, x)p(y, z) \equiv (Qxz \vee (x = z \wedge Qwy)))[\xi, a, a', b, b']$, so $(A, \rho) \vDash \textsc{lexord}(p, S, S')[\xi, \chi, \chi']$. This leads to the following result:

Lemma 5.2. *Where (A, ρ) is a strict well-order,*
$(A, \rho) \vDash \textsc{lexord}(p, S, S')[\xi, \chi, \chi']$ *iff* $\xi \restriction (\chi \times \chi')$ *is a homomorphism from* $(\chi, \rho \restriction \chi) \otimes (\chi', \rho \restriction \chi')$ *into* $(\xi[\chi \times \chi'], \rho \restriction \xi[\chi \times \chi'])$.

Next, $\textsc{oprod}(S, S', S'')$ is

$$\exists p(\exists x Sx \wedge \exists y S'y \wedge \textsc{pairs}(p, S, S', S'') \wedge \textsc{lexord}(p, S, S')) \tag{45}$$

Consider a strict well-order (A, ρ). By facts about $\textsc{pairs}(p, S, S', S'')$ and $\textsc{lexord}(p, S, S')$ noted above, $(A, \rho) \vDash \textsc{oprod}(S, S', S'')[\chi, \chi', \chi'']$ iff there is ξ, a binary function on A, such that $\xi \restriction (\chi \times \chi')$ is a bijection from $\chi \times \chi'$ to χ'', and $\xi \restriction (\chi \times \chi')$ is a homomorphism from $(\chi, \rho \restriction \chi) \otimes (\chi', \rho \restriction \chi')$ into $(\xi[\chi \times \chi'], \rho \restriction \xi[\chi \times \chi'])$, thus:

Lemma 5.3. *For any strict well-order (A, ρ),*
$(A, \rho) \vDash \textsc{oprod}(S, S', S'')[\chi, \chi', \chi'']$ *iff* $(\chi'', \rho \restriction \chi'')$ *is isomorphic to* $(\chi, \rho \restriction \chi) \otimes (\chi', \rho \restriction \chi')$.

Now, given sentences φ and ψ, $\varphi \bullet \psi$ is

$$\textsc{swo} \wedge \exists S S' S''(\textsc{oprod}(S, S', S'') \wedge \varphi[S] \wedge \psi[S'] \wedge \textsc{dom}(S'')) \tag{46}$$

If $(A, \rho) \vDash \varphi \bullet \psi$, then (A, ρ) is a strict well-order and there are χ and χ', nonempty subsets of A, such that (A, ρ) is isomorphic to $(\chi, \rho \upharpoonright \chi) \otimes (\chi', \rho \upharpoonright \chi')$, $(\chi, \rho \upharpoonright \chi) \vDash \varphi$, and $(\chi', \rho \upharpoonright \chi') \vDash \psi$. Thus, there are $\alpha, \beta > 0$ such that (A, ρ) is isomorphic to $(\alpha \bullet \beta, \in)$, $(\alpha, \in) \vDash \varphi$, and $(\beta, \in) \vDash \psi$. On the other hand, consider $\alpha, \beta > 0$ such that $(\alpha, \in) \vDash \varphi$ and $(\beta, \in) \vDash \psi$. Note that $\alpha \subseteq \alpha \bullet \beta$ and $\beta \subseteq \alpha \bullet \beta$, and that $(\alpha \bullet \beta, \in) \vDash \textsc{oprod}(S, S', S'')[\alpha, \beta, \alpha \bullet \beta]$, so $(\alpha \bullet \beta, \in) \vDash \varphi \bullet \psi$, so, for any (A, ρ) isomorphic to $(\alpha \bullet \beta, \in)$, $(A, \rho) \vDash \varphi \bullet \psi$. This establishes the following:

Lemma 5.4. *$(A, \rho) \vDash \varphi \bullet \psi$ iff there are $\alpha, \beta > 0$ such that (A, ρ) is isomorphic to $(\alpha \bullet \beta, \in)$, $(\alpha, \in) \vDash \varphi$, and $(\beta, \in) \vDash \psi$.*

The above makes it possible to show that the finitely characterizable ordinals are closed under ordinal addition and multiplication:

Theorem 5.1. *If $(\alpha, \in)$ and $(\beta, \in)$ are finitely characterizable (characterized by φ_α and φ_β, respectively), then so are $(\alpha + \beta, \in)$ and $(\alpha \bullet \beta, \in)$ (characterized by $\varphi_\alpha + \varphi_\beta$ and $\varphi_\alpha \bullet \varphi_\beta$, respectively).*

Proof. Suppose that φ_α characterizes $(\alpha, \in)$ and φ_β characterizes $(\beta, \in)$. It follows that if $(A, \rho) \vDash \varphi_\alpha + \varphi_\beta$ then (A, ρ) is isomorphic to $(\gamma + \delta, \in)$ for some $\gamma, \delta > 0$, such that $(\gamma, \in) \vDash \varphi_\alpha$ and $(\delta, \in) \vDash \varphi_\beta$, so $\gamma = \alpha$ and $\delta = \beta$, so (A, ρ) is isomorphic to $(\alpha + \beta, \in)$. On the other hand, $(\alpha + \beta, \in) \vDash \varphi_\alpha + \varphi_\beta$, $\alpha, \beta > 0$, $(\alpha, \in) \vDash \varphi_\alpha$ and $(\beta, \in) \vDash \varphi_\beta$. Thus $(A, \rho) \vDash \varphi_\alpha + \varphi_\beta$ iff (A, ρ) is isomorphic to $(\alpha + \beta, \in)$. The same argument can be used for $\alpha \bullet \beta$, substituting multiplication for addition and $\varphi_\alpha \bullet \varphi_\beta$ for $\varphi_\alpha + \varphi_\beta$. ■

This approach appears not to yield an analogous result for infinitely characterizable ordinals, but a weaker closure result can be shown:

Theorem 5.2. *If $(\alpha, \in)$ is finitely characterizable and $(\beta, \in)$ is infinitely characterizable, then $(\alpha + \beta, \in)$ is infinitely characterizable.*

Proof. Let φ be a sentence that characterizes $(\alpha, \in)$, and suppose that $\gamma > 0$ and $\gamma \neq \alpha + \beta$. If $\gamma < \alpha$, then note that $(\alpha + \beta, \in) \vDash \exists S(\exists x Sx \wedge \varphi[S])$, whereas $(\gamma, \in) \nvDash \exists S(\exists x Sx \wedge \varphi[S])$. If $\gamma = \alpha$, then $(\alpha + \beta, \in) \vDash \neg\varphi$, while $(\gamma, \in) \nvDash \neg\varphi$. Finally, if $\gamma > \alpha$, then $\gamma = \alpha + \delta$ for some $\delta \neq \beta$. Since $(\beta, \in)$ is infinitely characterizable, it follows that there is ψ such that $(\beta, \in) \vDash \psi$ but $(\delta, \in) \nvDash \psi$. In this case, $(\alpha + \beta, \in) \vDash \varphi + \psi$, but $(\alpha + \delta, \in) \nvDash \varphi + \psi$, since otherwise there would have to be some δ' such that $\alpha+\delta = \alpha+\delta'$, and $(\delta', \in) \vDash \psi$, but it is a well-known algebraic property of ordinal addition that if $\alpha + \delta = \alpha + \delta'$, then $\delta = \delta'$, and ψ was picked such that $(\delta, \in) \nvDash \psi$, this yields a contradiction, so in fact it must be that $(\alpha + \delta, \in) \nvDash \varphi + \psi$. Thus, for any $\gamma > 0$ such that $\gamma \neq \alpha + \beta$, there is a sentence that distinguishes between $(\gamma, \in)$ and $(\alpha + \beta, \in)$, so $(\alpha + \beta, \in)$ is infinitely characterizable. ■

Theorem 5.3. *If* $(\alpha, \in)$ *is finitely characterizable and* $(\beta, \in)$ *is infinitely characterizable, then* $(\alpha \bullet \beta, \in)$ *is infinitely characterizable.*

Proof. Let φ be a sentence that characterizes $(\alpha, \in)$, and consider $\gamma > 0$ such that $\gamma \neq \alpha \bullet \beta$. Suppose that there is no $\delta > 0$ such that $\gamma = \alpha \bullet \delta$, it follows that $(\gamma, \in) \nvDash \exists S S' S'' (\textit{OPROD}(S, S', S'') \wedge \varphi[S] \wedge \textit{DOM}(S''))$, but since φ characterizes α it follows that $(\alpha \bullet \beta, \in) \vDash \exists S S' S'' (\textit{OPROD}(S, S', S'') \wedge \varphi[S] \wedge \textit{DOM}(S''))$. On the other hand, suppose there is such a δ. It must be the case that $\delta \neq \beta$, so, since β is infinitely characterizable, let ψ be a sentence such that $(\beta, \in) \vDash \psi$ but $(\delta, \in) \nvDash \psi$; it follows that $(\alpha \bullet \beta, \in) \vDash \phi \bullet \psi$ and $(\alpha \bullet \delta, \in) \nvDash \phi \bullet \psi$. Thus $(\alpha \bullet \beta, \in)$ is infinitely characterizable. ■

Note that the proofs of both Theorem 5.2 and the part of Theorem 5.1 that concerns addition also work for finite and infinite characterizability in $L_{2M(Q)}$.

A more general approach to the characterization of ordinal sums and products is possible. The techniques used above can, with slight modification, provide a formula $\textit{OSUM}(S, R, S', R', S'', R'')$ with the property that $\mathfrak{A} \vDash \textit{OSUM}(S, R, S', R', S'', R'')[\chi, \rho, \chi', \rho', \chi'', \rho'']$ iff $(\chi, \rho \restriction \chi)$, $(\chi', \rho' \restriction \chi')$, and $(\chi'', \rho'' \restriction \chi'')$ are all nonempty strict well-orders, and $(\chi'', \rho'' \restriction \chi'')$ is isomorphic to $(\chi, \rho \restriction \chi) \oplus (\chi', \rho' \restriction \chi')$. A formula $\textit{OPROD}$(S,R,S',R',S",R") with analogous behavior can also be constructed. Such formulas could be used to prove the closure results above, and would have a variety of other applications (for example, they could be used to show that if $\aleph_\alpha$ and $\aleph_\beta$ are both finitely characterizable, then so is $\aleph_{\alpha+\beta}$). This approach was not used here because because it cannot be applied to the monadic case, and because it would have made the formulas and proofs involved more cumbersome.

6. Relating Different Kinds of Characterizability

There are many interesting notions of characterizability besides those discussed above, and many important questions about the connections between different kinds of characterizability. A few of the relevant ideas and results are discussed below.

The notion of *characterizability in* plays an important part in the proofs in Section 2. A structure $\mathfrak{A}$ of signature $\mathbf{k}$ is characterizable in another structure $\mathfrak{B}$ of signature $\mathbf{k}'$ iff, for some variables $\vec{v}_{\mathbf{k}}$ of types corresponding to $\mathbf{k}$, there is a formula $\varphi(S, \vec{v}_{\mathbf{k}})$ of $L_{2(\mathbf{k}')}$ such that, for $\chi \subseteq B$ and $\vec{\vartheta}_{\mathbf{k}}$ values for $\vec{v}_{\mathbf{k}}$, $\mathfrak{B} \models \varphi[\chi, \vec{\vartheta}_{\mathbf{k}}]$ iff $(\chi, \vec{\vartheta}_{\mathbf{k}} \restriction \chi)$ is isomorphic to $\mathfrak{A}$. This notion is implicit in the proof in [11] that Dedekind algebras lack the second-order Löwenheim-Skolem property. For any finitely characterizable $\mathfrak{A}$ such that $card(A) \leq card(B)$, $\mathfrak{A}$ is characterizable in (B), but little else is known about the relationship between this notion of characterizability and others.

The notion of cardinal describability (defined in Section 1) has an intuitive connection to the notion of characterizability in a structure, but, as noted above, is equivalent to finite characterizability for nonzero cardinals. To see this, consider a cardinal $\kappa > 0$: if κ is characterized by φ, then clearly it is described by the relativized formula $\varphi[S]$, and if κ is described by some formula $\psi(S)$, then it is characterized by the formula $\exists S(\psi(S) \wedge \textsc{dom}(S))$.

In [3] and [4], the primary notion of characterizability used is that of finite $\Diamond^1_2$-characterizability. Although this notion of characterizability has some idiosyncracies, it is closely connected with second-order finite characterizability: as is pointed out in [3] and [4] (Theorem 2.2 and Corollary 2.3 of the latter), if $\mathscr{S}$ is a second-order spectrum which contains no finite cardinal, then $\{(2^\kappa)^+ : \kappa \in \mathscr{S}\}$ is a $\Diamond^1_2$-spectrum. Thus, if κ is second-order finitely characterizable then $(2^\kappa)^+$ is finitely $\Diamond^1_2$-characterizable.

Turning again to finite and infinite characterizability, it is natural to ask whether, at least in the case of cardinals and ordinals, the two notions are coextensive. It is easy to see that finite characterizability always implies infinite characterizability, but it is possible to show that there are infinitely characterizable structures that are not finitely characterizable. Indeed, it follows from comments in [10] that there are infinitely characterizable Dedekind algebras the cardinality of which is not finitely characterizable (since there is an infinitely characterizable Dedekind algebra the range whose configuration signature is the set of finitely characterizable cardinals). This calls attention to how different infinite and finite characterizability can be, but says nothing about the case of cardinals in particular. Even in the case of cardinals, this question has implications for other issues in second-order model theory. For example, it is shown in [9] that it is equivalent to the question of whether there are homogeneous-universal Dedekind algebras that are (infinitely) characterizable but not finitely characterizable.

Although I am not aware of any proof that finite and infinite cardinal characterizability are distinct, a weaker result is sufficient to motivate the independent study of both notions. Observe that there are exactly $\aleph_0$ finitely characterizable ordinals. In [1] it is shown that it follows from the axiom of constructibility (and so is consistent with ZFC) that no two non-isomorphic countable relational structures of finite signature are second-order indistinguishable. Since $\aleph_0$ is a finitely characterizable cardinal, this entails that each countable relational structure is distinguishable from all other relational structures, so the axiom of constructibility implies that for every countable ordinal $\alpha > 0$, $(\alpha, \in)$ is infinitely characterizable. There are $\aleph_1$ distinct countable ordinals, thus:

Proposition 6.1. *It is consistent with ZFC that the cardinality of the set of infinitely characterizable ordinals be greater than the cardinality of the set of finitely characterizable ordinals.*

This means that it is consistent with ZFC that there be infinitely characterizable ordinals that are not finitely characterizable.

Corollary 2.2 makes it possible to extend Proposition 6.1 to the cardinals. Since the axiom of constructibility implies that for every countable ordinal $\alpha > 0$, $(\alpha, \in)$ is infinitely characterizable, and by Corollary 2.2 this entails that for every countable ordinal $\alpha > 0$, $\aleph_\alpha$ is infinitely characterizable. So if there are at least $\aleph_1$ distinct countable ordinals, it follows that there are at least $\aleph_1$ distinct infinitely characterizable cardinals, but there are only $\aleph_0$ finitely characterizable cardinals. This proves the following:

Proposition 6.2. *It is consistent with ZFC that the cardinality of the set of infinitely characterizable cardinals be greater than the cardinality of the set of finitely characterizable cardinals.*

As above, it follows immediately that the existence of cardinals that are infinitely but not finitely characterizable is consistent with ZFC.

7. Open Problems

This paper has presented a variety of results about finitely and infinitely characterizable cardinals and ordinals. Many of these results raise questions that, to the knowledge of the author, are still open. A list of some of the more significant ones appears below.

First, is the converse of Corollary 2.1 or Corollary 2.2 true? An affirmative answer for both would eliminate the need for separate studies of cardinal and ordinal characterizability, and an affirmative answer for either would simplify matters significantly. Negative answers would point to the existence of an $\aleph_\alpha$ that was more characterizable than the associated $(\alpha, \in)$. The nature of this α would be of considerable interest, and would lead to a richer understanding of the expressive resources of second-order logic.

Second, what precisely is the relationship between finite and infinite cardinal characterizability? It was shown in Section 6 that it is consistent with ZFC that finite and infinite cardinal (and ordinal) characterizability be distinct, but it is not immediately clear whether this state of affairs follows from ZFC or is independent of it. If finite and infinite characterizability are not coextensive, it is natural to ask whether some subtler connection exists. It is also interesting to ask if there is *any* nontrivial class of structures for which finite and infinite second-order characterizability are equivalent, and, if so, what characteristic distinguishes those classes of structures for which this equivalence holds.

Third, although results here and elsewhere have done much to map the shape of the sets of finitely and infinitely characterizable cardinals, much remains unknown. In particular, the author is not aware of any proofs or disproofs of the converses of Propositions 4.6 and 4.7. Thus, it is not even known whether there is an uncharacterizable cardinal with a finitely characterizable successor. If the existence of such a cardinal were known, it would considerably complicate the theory of characterizability for many kinds of structures: in particular, because of Propostion 5.3, it would entail that there was a finitely characterizable cardinal $\aleph_\alpha$ such that the corresponding $(\alpha, \in)$ was not finitely characterizable. Of course, much more intricate questions might be asked, and many of these also remain open.

Fourth, what other classes of structures have the second-order Fraenkel-Carnap property? Unstructured sets were shown to have the Fraenkel-Carnap property in [2], and [11] and [12] established this property for a number of other relatively simple classes of structures. Some extensions of these results exist: it is easy to apply the approach used in [11] to show that for any $\mathbf{k}$ that contains only individual and unary predicate constants, the models of $L_{2(\mathbf{k})}$ have the Fraenkel-Carnap property, and Irena Penev has extended the arguments from Section 3 to cover well-orders with distinguished elements. There is also some reason to hope that more progress can be made if attention is restricted to countable models. It is interesting to note that all the proofs mentioned above exploit either a well-ordering on the domain of a model, or a well-ordering of cardinals, suggesting a more general applicability of the notion of well-ordering to the Fraenkel-Carnap question. In spite of everything said above, for almost all mathematically interesting structures we are entirely in the dark. Indeed, in light of the present state of our knowledge, it is conceivable that all structures have the second-order Fraenkel-Carnap property. An affirmative answer here would provide a reasonably deep, significant, and satisfying result in second-order model theory; a proof of a negative answer would also be quite enlightening.

Fifth, although monadic second-order logic receives only secondary attention above, the results are interesting enough to encourage further study. In particular, the techniques used to establish most of the Fraenkel-Carnap properties cited above will not work in monadic second-order logic, but, as has been seen, at least one class of structures does have a nontrivial monadic second-order Fraenkel-Carnap property. Are there others, and, if so, what are they? In addition, although some of the results concerning ordinal arithmetic were shown for the monadic case, many ordinal operations require further study in the monadic case.

Finally, what is the natural generalization of the "closure" results in Sections 4 and 5? That is, where Δ is a class of structures, and Ξ is an n-ary (partial) function (in the sense that a suitable proper class of ordered tuples can understood as a function) on Δ, what conditions will guarantee that Ξ preserves finite (or infinite) characterizability? A reasonably general partial answer to this question would give us new ways to apply existing structural and algebraic results to the study of second-order characterizability.

Acknowledgements. This paper is the result of my work as a research assistant to Professor George Weaver at Bryn Mawr College. I would like to thank Professor Weaver for his guidance, support and patience. I am also indebted both to him and to Irena Penev for listening to much less coherent versions of some of these arguments and offering many useful critiques and suggestions, and for bringing many relevant papers and results to my attention. Finally, I gratefully acknowledge Jolanta Monikowska and Melvin Fitting for their editorial work, and Chaos Golubitsky, Elliot Reed, Erik Osheim, and Josh Burdick for their invaluable help with the proofreading process; what errors remain are mine alone.

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